HOSSAM GHANEM

(17) 3.2 Definition of Derivative (A)

$$f^{\backslash}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 1 52 April 9, 2009 A	State the definition of the derivative of the function f at $x = a$.

Solution

$$f^{\backslash}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f^{\backslash}(a) = \lim_{h \to 0} \frac{f(h+a) - f(a)}{h}$$

Example 2	Show that if f is differentiable at $x = a$. Then f is continuous at $x = a$

Solution

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a)$$

$$f(x) = f(a) + \frac{f(x) - f(a)}{x - a}(x - a)$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} f(a) + \frac{f(x) - f(a)}{x - a}(x - a)$$

f is differentiable

$$\lim_{x \to a} f(x) = f(a) + \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a)$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f^{\backslash}(x)$$

$$\lim_{x \to a} f(x) = f(a) + f^{\backslash}(x) \cdot 0$$

$$\lim_{x \to a} f(x) = f(a)$$

Then f is continuous at $x = a$

Example 3	Evaluate: $\lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x^{\frac{1}{2}}-2^{\frac{1}{2}}}$ If $f^{\backslash}(2)=4$

Solution

$$L = \lim_{x\to 2} \frac{f(x)-f(2)}{\sqrt{x}-\sqrt{2}} = \lim_{x\to 2} \frac{(f(x)-f(2))(\sqrt{x}+\sqrt{2})}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})} = \lim_{x\to 2} \frac{f(x)-f(2)}{(x-2)} \cdot \lim_{x\to 2} \sqrt{x}+2$$

$$= f^{\backslash}(2)\cdot(\sqrt{2}+\sqrt{2}) = 4\cdot(2\sqrt{2}) = 8\sqrt{2}$$

Example 4	Find the limit if it exists $\lim\limits_{h\to 0} \dfrac{(x+h)^3-x^3}{h}$ 25 January 12 .2003

Solution

$$L = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$$

Let $f(x) = x^3$ $\therefore$ $f(x+h) = (x+h)^3$ & $f^{\backslash}(x) = 3x^2$

$$L = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f^{\backslash}(x) = 3x^2$$

Example 5 33 October 25, 2001 A	Write the definition of the derivative of f at 1. Then use it to find $f^{\backslash}(1)$ if it exists, where $f(x)=\sqrt{x^2-2x+1}$ Justify your answer.

Solution

$$f^{\backslash}(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$$

$f(x) = \sqrt{x^2-2x+1}$

$f(1) = \sqrt{1-2+1} = 0$

$$f^{\backslash}(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{\sqrt{x^2-2x+1}-0}{x-1} = \lim_{x\to 1} \frac{\sqrt{(x-1)^2}}{x-1} = \lim_{x\to 1} \frac{|x-1|}{x-1}$$

$$L_1 = \lim_{x\to 1^+} \frac{|x-1|}{x-1} = \lim_{x\to 1^+} \frac{(x-1)}{x-1} = 1$$

$$L_2 = \lim_{x\to 1^-} \frac{|x-1|}{x-1} = \lim_{x\to 1^-} \frac{-(x-1)}{x-1} = -1$$

$$\therefore \lim_{x\to 1} \frac{|x-1|}{x-1} \quad D.N.E$$

$$f^{\backslash}(1) \quad D.N.E$$

Example 6 13 November 1995	Use the definition of derivative to find $f^{\backslash}(x)$, where $f(x) = \dfrac{1}{x-1}$ where $x \neq 1$

Solution

$$f^{\backslash}(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} = \lim_{h\to 0} \frac{(x-1)(x+h-1)\left(\frac{1}{x+h-1} - \frac{1}{x-1}\right)}{(x-1)(x+h-1)h}$$

$$= \lim_{h\to 0} \frac{(x-1)-(x+h-1)}{h(x-1)(x+h-1)} = \lim_{h\to 0} \frac{x-1-x-h+1}{h(x-1)(x+h-1)} = \lim_{h\to 0} \frac{-h}{h(x-1)(x+h-1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x-1)(x+h-1)} = \lim_{h\to 0} \frac{-1}{(x-1)(x+0-1)} = \lim_{h\to 0} \frac{-1}{(x-1)(x-1)} = \frac{-1}{(x-1)^2}$$

Example 7 60 *October* 31 , 2011	(4 points) Let $f(x) = \sqrt{x+1}$. Use the definition of derivative to find $f'(3)$.

Solution

$$f(x) = \sqrt{x+1}$$

$$f(3) = \sqrt{3+1} = 2$$

$$f^{\backslash}(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} = \lim_{x\to 3} \frac{\sqrt{x+1} - 2}{x-3} = \lim_{x\to 3} \frac{(\sqrt{x+1} - 2)(\sqrt{x+1} + 2)}{(x-3)(\sqrt{x+1} + 2)} = \lim_{x\to 3} \frac{x+1-4}{(x-3)(\sqrt{x+1} + 2)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x+1} + 2)} = \lim_{x\to 3} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{\sqrt{3+1} + 2} = \frac{1}{4}$$

Homework

1 47 November 10, 2007 A	If $f^{\backslash}(2) = -8$ then find $\lim_{x \to 2} \frac{f(x) - f(2)}{x^3 - 8}$
2 50 November 17, 2008 A	Let f be a function such that $f^{\backslash}(1) = 3$. Find $\lim_{x \to 1} \frac{f(x) - f(1)}{x^2 - 1}$
3	Let $f(x) = 2x^2 - 5x + 1$ Using the definition of the derivative, find $f^{\backslash}(1)$
4 25 December 10, 2000	Let $f(x) = x^2 + x - 2$ .Use the definition of the derivative to find f$^{\backslash}$(1)
5 54 November 16, 2009 A	Use the definition of the derivative to find $f^{\backslash}(2)$, where $f(x) = \sqrt{x+2}$
6 34 March 23, 2002	Use the definition of derivative to find $f^{\backslash}(1)$ where $f(x) = \sqrt{5 - 4x}$
7 40 October 28, 2004 A	Use the definition of derivative to find $f^{\backslash}(3)$ where $f(x) = \sqrt{x+1}$
8 35 October 31, 2002 A	Let $f(x) = \sqrt{7 - 3x}$ Use the definition of derivative to find $f^{\backslash}(1)$
9 45 March 28, 2007	Use the definition of the derivative to find $f^{\backslash}(0)$if $f(x) = \frac{1}{\sqrt{x+1}}$
10 53 July 18, 2009 A	Use the definition of the derivative to find $f^{\backslash}(1)$, where $f(x) = \frac{1}{\sqrt{x}} + 1$
11	Use the definition of derivative to find $f^{\backslash}(x)$, where $f(x) = \frac{1}{x+2}$, $x \neq 1$

Homework

12 48 March 25, 2008 A	Use the definition of derivative to find $f^{\backslash}(2)$ where $f(x)=\sqrt{x+2}-1$
13 14 March 1996	Use the definition of derivative to find $\frac{ds}{dt}$, where $s=t^2+1$
14 18 May 24 ,2000	Use the definition of the derivative to find $f'(1)$, where $f(x)=\sqrt{3x+1}$
15 58 7April 2011	[3 pts.] The following limit represents f\(a) for a function f and a number a. Identify fand a , and use this information to compute the limit . $\lim_{x\to-1}\frac{x^{100}-1}{x+1}$
16 55 April 8, 2010	(3pts) Use the definition of the derivative to evaluate $f^{\backslash}(-1)$, for $f(x)=\sqrt{1-x}$
17 56 July 10, 2010	Use the definition of the derivative to find $f^{\backslash}(x)$ if $f(x)=\sqrt{x+5}$ (3 points)
18 57 November 8, 2010	Use the definition of the derivative to show that f is differentiable at -1, where $f(x)=\begin{cases}-2x-1 & ,\text{if } x\le -1\\ x^2 & ,\text{if } x>-1\end{cases}$ (3 pts.)
19 26 May 10, 2001	Use the definition of derivative to find $f^{\backslash}(1)$ if $f(x)=\sqrt[3]{x}$

Homework

20 2 November 1989	Let $f(x) = x^2 + 3x + 4$ Using the definition of the derivative, find $f^{\backslash}(1)$
21 32 August 02, 2008	Use the definition of derivative to find $f^{\backslash}(1)$ where $f(x) = x + \sqrt{x} - 1$
22 38 March 31, 2004	Use the definition of derivative to find $f^{\backslash}(1)$ where $f(x) = \dfrac{1}{1+\sqrt{x}}$
23 41 March 30, 2005	Use the definition of derivative to find $f^{\backslash}(2)$ where $f(x) = \sqrt{x^3 + 1}$
24 35 August 15, 2009	Let $f(x) = 2x + \lvert x \rvert$ (a) Find the points , if any , where f is discontinuous . Justify your answer. (b) Find the points , if any ,where f is not differentiable . Justify your answer

20 2 November 1989	Let $f(x) = x^2 + 3x + 4$ Using the definition of the derivative, find $f^{\backslash}(1)$

Solution

$f(x) = x^2 + 3x + 4$

$f(1) = 1 + 3 + 4 \ = 8$

$$f^{\backslash}(a) = \lim_{h \to a} \frac{f(x) - f(a)}{x - a}$$

$$f^{\backslash}(1) = \lim_{x \to 1} \frac{x^2 + 3x + 4 - 8}{x - 1} = \lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 4)}{x - 1} = \lim_{x \to 1} (x + 4) = 5$$

21 32 August 02, 2008	Use the definition of derivative to find $f^{\backslash}(1)$ where $f(x) = x + \sqrt{x} - 1$

Solution

$f(x) = x + \sqrt{x} - 1$

$f(1) = 1 + 1 - 1 = 1$

$$f^{\backslash}(a) = \lim_{h \to a} \frac{f(x) - f(a)}{x - a}$$

$$f^{\backslash}(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x + \sqrt{x} - 1 - 1}{x - 1} = \lim_{x \to 1} \frac{x + \sqrt{x} - 2}{x - 1} = \lim_{x \to 1} \frac{(x - 2) + \sqrt{x}}{x - 1}$$

$$= \lim_{x \to 1} \frac{[(x - 2) + \sqrt{x}][(x - 2) - \sqrt{x}]}{(x - 1)[(x - 2) - \sqrt{x}]} = \lim_{x \to 1} \frac{(x - 2)^2 - x}{(x - 1)[(x - 2) - \sqrt{x}]} = \lim_{x \to 1} \frac{x^2 - 4x + 4 - x}{(x - 1)[(x - 2) - \sqrt{x}]}$$

$$= \lim_{x \to 1} \frac{x^2 - 5x + 4}{(x - 1)[(x - 2) - \sqrt{x}]} = \lim_{x \to 1} \frac{(x - 4)(x - 1)}{(x - 1)[(x - 2) - \sqrt{x}]} = \lim_{x \to 1} \frac{(x - 4)}{(x - 2) - \sqrt{x}} = \frac{-3}{-1 - 1} = \frac{3}{2}$$

22 38 March 31, 2004	Use the definition of derivative to find $f^{\backslash}(1)$ where $f(x) = \frac{1}{1 + \sqrt{x}}$

Solution

$$f(x) = \frac{1}{1 + \sqrt{x}}$$

$$f(1) = \frac{1}{1 + \sqrt{1}} = \frac{1}{2}$$

$$f^{\backslash}(a) = \lim_{h \to a} \frac{f(x) - f(a)}{x - a}$$

$$f^{\backslash}(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{1 + \sqrt{x}} - \frac{1}{2}}{x - 1} = \lim_{x \to 1} \frac{2(1 + \sqrt{x})\left(\frac{1}{1 + \sqrt{x}} - \frac{1}{2}\right)}{2(1 + \sqrt{x})(x - 1)}$$

$$= \lim_{x \to 1} \frac{2 - (1 + \sqrt{x})}{2(1 + \sqrt{x})(x - 1)} = \lim_{x \to 1} \frac{1 - \sqrt{x}}{2(1 + \sqrt{x})(x - 1)} = \lim_{x \to 1} \frac{-(\sqrt{x} - 1)}{2(1 + \sqrt{x})(\sqrt{x} - 1)(\sqrt{x} + 1)}$$

$$= \lim_{x \to 1} \frac{-1}{2(1 + \sqrt{x})^2} = \frac{-1}{2(2)^2} = \frac{-1}{8}$$

23 41 March 30, 2005	Use the definition of derivative to find $f^{\backslash}(2)$ where $f(x) = \sqrt{x^3+1}$

Solution

$f(x) = \sqrt{x^3+1}$

$f(2) = \sqrt{8+1} = \sqrt{9} = 3$

$$f^{\backslash}(a) = \lim_{h\to a} \frac{f(x)-f(a)}{x-a}$$

$$f^{\backslash}(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{\sqrt{x^3+1}-3}{x-2} = \lim_{x\to 2} \frac{\left(\sqrt{x^3+1}-3\right)\left(\sqrt{x^3+1}+3\right)}{(x-2)\left(\sqrt{x^3+1}+3\right)}$$

$$= \lim_{x\to 2} \frac{x^3+1-9}{(x-2)\left(\sqrt{x^3+1}+3\right)} = \lim_{x\to 2} \frac{x^3-8}{(x-2)\left(\sqrt{x^3+1}+3\right)}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)\left(\sqrt{x^3+1}+3\right)} = \lim_{x\to 2} \frac{(x^2+2x+4)}{\left(\sqrt{x^3+1}+3\right)} = \frac{4+4+4}{3+3} = \frac{12}{6} = 2$$

24 35 August 15, 2009	Let $f(x) = 2x + \lvert x\rvert$ (a) Find the points , if any , where f is discontinuous . Justify your answer. (b) Find the points , if any ,where f is not differentiable . Justify your answer

Solution

$$f(x) = \begin{cases} 2x+x & \text{If } x>0 \\ 0 & \text{If } x=0 \\ 2x-x & \text{If } x<0 \end{cases} \quad \rightarrow \quad f(x) = \begin{cases} 3x & \text{If } x>0 \\ 0 & \text{If } x=0 \\ x & \text{If } x<0 \end{cases}$$

$f(x) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x = 0$

$\therefore \lim_{x\to 0} f(x) = 0 = f(0)$

$\therefore$ f is cont. on R

$$\therefore f^{\backslash}(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x-0}{x-0} = 1$$

$$\therefore f^{\backslash}(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{3x}{x} = 3$$

$f^{\backslash}(0^-) \neq f^{\backslash}(0^+)$

$\therefore$ f *is* not differentiable *at* $x = 0$